A Tight Upper Bound for the Third-Order Asymptotics for Most Discrete Memoryless Channels

Marco Tomamichel[1] and Vincent Y. F. Tan[2,3]

[1]Centre for Quantum Technologies, National University of Singapore, Singapore
[2]Institute for Infocomm Research (I^{2}R), A*STAR
[3]Department of Electrical and Computer Engineering, National University of Singapore, Singapore

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- Φ is the cumulative (normal) Gaussian distribution.

Known Results

- Shannon's noisy coding theorem and Wolfowitz's strong converse imply

$$\lim_{n\to\infty} \frac{1}{n} \log M^*(W^n, \varepsilon) = C \quad \text{bits/channel use}$$

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  where $C := \max_P I(P, W)$ is the channel capacity.
- Finer asymptotic characterizations are desirable to understand the backoff from capacity for finite n.

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$$V_\varepsilon := \begin{cases} V_{\min} & \text{if } \varepsilon < \frac{1}{2} \\ V_{\max} & \text{if } \varepsilon \geq \frac{1}{2} \end{cases}, \quad \text{where}$$

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- Strassen's Gaussian approximation states that

$$\log M^*(W^n, \varepsilon) = nC + \sqrt{nV_\varepsilon}\Phi^{-1}(\varepsilon) + \rho_n,$$

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- See also Hayashi, using information spectrum methods.

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where $\rho_n = o(\sqrt{n})$.

- See also Hayashi, using information spectrum methods.
- In fact, $O(1) \leq \rho_n \leq O(\log n)$ unless $\varepsilon > \frac{1}{2}$ and the channel is exotic as pointed out by Polyanskiy-Poor-Verdú (PPV).

Known Results

- Polyanskiy showed the following further refinements.
- Lower bound (achievability): If W has positive reverse dispersion (e.g. all entries of W positive) then

$$\rho_n \geq \log \sqrt{n} + O(1).$$

(The reverse dispersion is $V(PW, W^{-1})$, where P achieves the Gaussian approximation and W^{-1} is the inverse channel.)

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- Our contribution: The above converse bound holds **for all** DMC's (unless $\varepsilon > \frac{1}{2}$ and W is exotic).

Main Result

Theorem

For every DMC W and ε with $V_\varepsilon > 0$, the blocklength n, ε-error capacity satisfies

$$\log M^*(W^n, \varepsilon) \le nC + \sqrt{nV_\varepsilon}\Phi^{-1}(\varepsilon) + \log\sqrt{n} + O(1).$$

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- This converse is tight in the third order, except for channels with vanishing reverse dispersion, e.g. the binary erasure channel.

One-Shot Converse

- A "symbol-wise" relaxation of PPV's meta-converse to a relative entropy information spectrum.

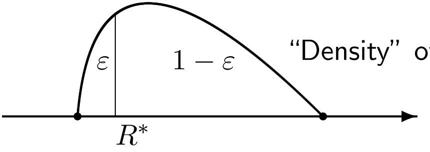
Proposition

For any $\delta \in (0, 1-\varepsilon)$, *we have*

$$\log M^*(W,\varepsilon) \le \inf_{Q \in \mathcal{P}(\mathcal{Y})} \sup_{x \in \mathcal{X}} D_s^{\varepsilon+\delta}(W(\cdot|x) \| Q) + \log \frac{1}{\delta},$$

- The relative entropy information spectrum is given as

$$D_s^{\varepsilon}(P\|Q) = \sup \left\{ R \in \mathbb{R} \,\middle|\, \Pr\left[\log \frac{P}{Q} \le R \right] \le \varepsilon \right\}.$$



"Density" of $\log \frac{P}{Q}$ under P

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- The bound trivially specializes to channels with cost constraints, where the supremum is taken over $x \in \Gamma \subseteq \mathcal{X}$.
- We will usually choose $\delta = \frac{1}{\sqrt{n}}$ so that the last term is $\log \sqrt{n}$.

Type-Counting Method

- For n repetition of the channel, we choose $\delta = \frac{1}{\sqrt{n}}$ and set $\varepsilon' := \varepsilon + \delta$. Then, the converse yields

$$\log M^*(W^n, \varepsilon) \leq \inf_{Q^{(n)}} \sup_{x^n} D_s^{\varepsilon'}(W^n(\cdot|x^n) \| Q^{(n)}) + \log \sqrt{n},$$

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- Hayashi uses a convex combination of "output types", e.g.

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- Then the converse relaxes to

$$\log M^*(W^n, \varepsilon) \leq \sup_{x^n} D_s^{\varepsilon'}(W^n(\cdot|x^n)\|(P_{x^n}W)^n) + \log\sqrt{n} + \log|\mathcal{P}_n(\mathcal{X})|,$$

where P_{x^n} is the type (empirical distribution) of x^n.

Type-Counting Method

- Using the Berry-Esseen theorem, it is easy to see that

$$\begin{aligned}&\sup_{x^n} D_s^{\varepsilon'}(W^n(\cdot|x^n)\|(P_{x^n}W)^n)\\&\qquad = \sup_{x^n} nI(P_{x^n}, W) + \sqrt{nV(P_{x^n}, W)}\Phi^{-1}(\varepsilon') + O(1)\\&\qquad \leq nC + \sqrt{nV_\varepsilon}\Phi^{-1}(\varepsilon) + O(1)\end{aligned}$$

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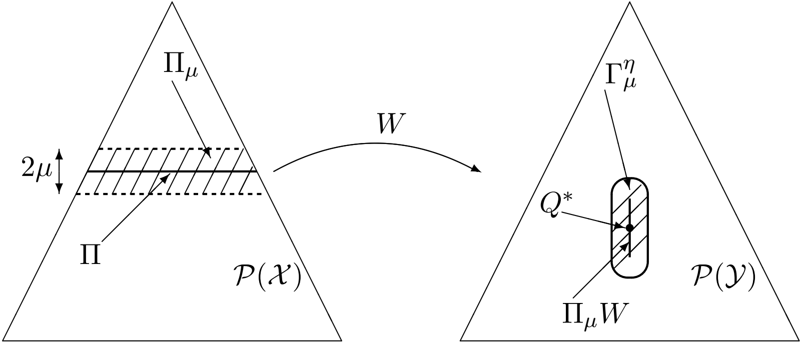
- We also have $\log|\mathcal{P}_n(\mathcal{X})| \leq (|\mathcal{X}|-1)\log n + o(1)$.
- Combining these upper bounds, we find

$$\log M^*(W^n, \varepsilon) \leq nC + \sqrt{nV_\varepsilon}\Phi^{-1}(\varepsilon) + \log\sqrt{n} + \big(|\mathcal{X}|-1\big)\log n + O(1).$$

- This recovers the converse bounds by PPV (and Strassen) for general DMCs.

Approximately Capacity-Achieving Types



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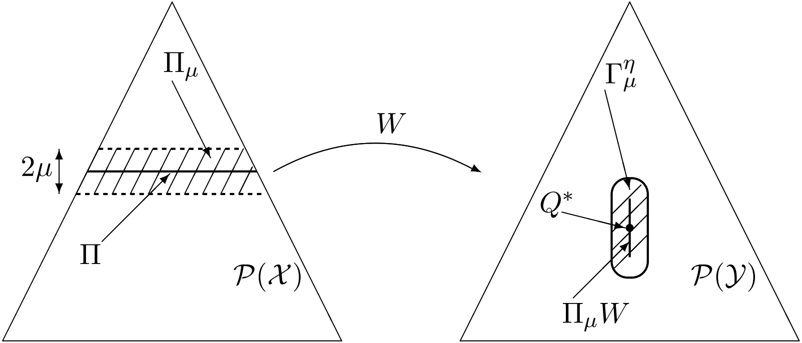

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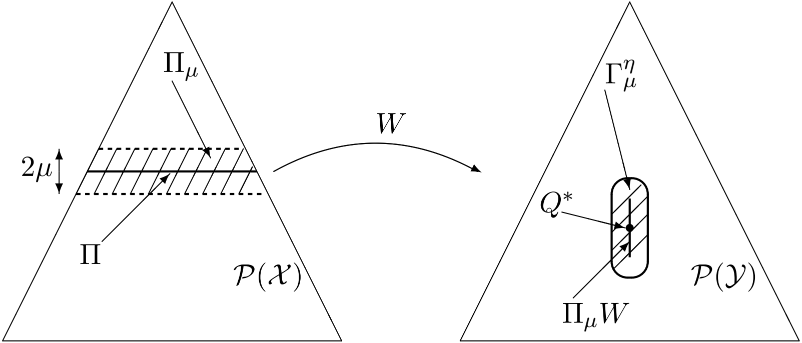

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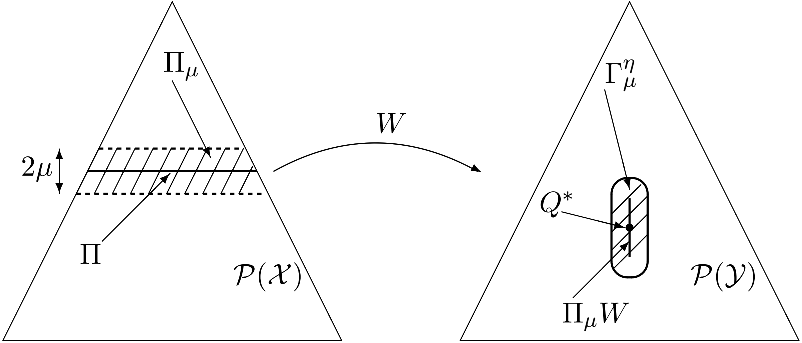

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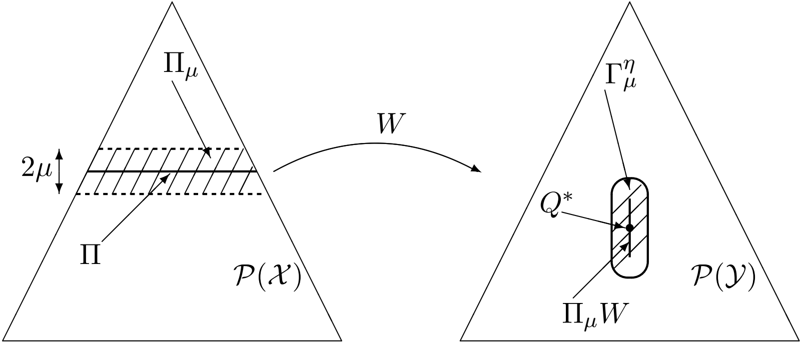

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- But we need a different technique for $P_{x^n} \in \Pi_\mu$!
- Our approach: Span an ϵ-net over Γ_μ^η.

An ϵ-Net of Output Distributions

- We choose the following output distribution:

$$Q^{(n)}(y^n) := \frac{1}{2} \sum_{P \in \mathcal{P}_n(\mathcal{X})} \frac{1}{|\mathcal{P}_n(\mathcal{X})|} \prod_{i=1}^{n} PW(y_i) + \frac{1}{2} \sum_{\mathbf{k} \in \mathcal{K}} \frac{\exp\left(-\gamma \|\mathbf{k}\|_2^2\right)}{F(n)} \prod_{i=1}^{n} Q_{\mathbf{k}}(y_i)$$

where $F(n)$ is used for normalization, $\gamma > 0$, and

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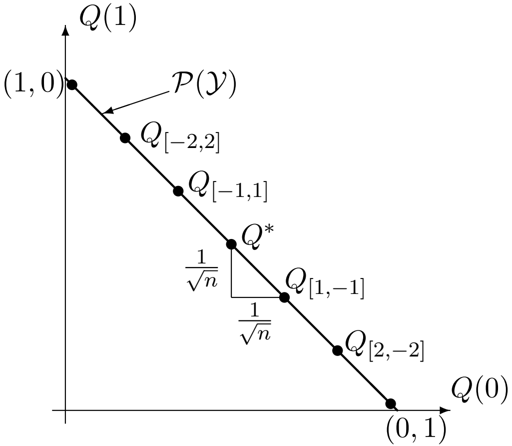
- The net get denser as n increases.
- Due to the exponential damping, $F(n)$ can be bounded by a constant independent of n.

An ϵ-Net of Output Distributions

- For every $P \in \mathcal{P}(\mathcal{X})$, $\exists\,\mathbf{k}$ with $\|Q_{\mathbf{k}} - PW\|_2 \le O(\frac{1}{\sqrt{n}})$.



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- Then, using various continuity properties

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&D_s^{\varepsilon'}(W^n(\cdot|x^n)\|Q^{(n)}) \\
&\quad \leq D_s^{\varepsilon'}(W^n(\cdot|x^n)\|Q_{\mathbf{k}}^n) + \gamma\|\mathbf{k}\|_2^2 + O(1) \\
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- We also find

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\|\mathbf{k}\|_2 &= \sqrt{n}\|Q_{\mathbf{k}} - Q^*\|_2 \\
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- This term can be compensated for since we have, for $x^n \in \Pi_\mu$,

$$
I(P_{x^n}, W) \le C - \alpha\|P_{x^n} - \Pi\|_2^2
$$

for some $\alpha > 0$.

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- We know that $\rho_n = O(1)$ for the binary erasure channel (see, e.g., PPV).
- It remains to find the third order term for other channels with vanishing reverse dispersion (and non-vanishing dispersion).